The Projection of Fourfold Figures upon a Three-Flat.

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1.—*Introductory.*

Stringham has shown* that in space of any dimensions there are at least three regular figures. These three series of figures may be developed synthetically from a point, each according to its own law of increase; and in such ways as this it is possible to develop a synthetic geometry of higher space along with, perhaps, such distinctness of conception as we now have of space of one, two and three dimensions.

Hinton,† indeed, maintains not only that such a conception of four-fold space is possible, but that it can be attained with comparative ease by a careful synthetic study of a few four-fold figures.

I have adopted as most convenient and complete the nomenclature in which the order of an n-fold solid is indicated by a word designating the number of its axes; as tessaract, pentact, etc.; and in which the particular solids of each order are distinguished by numeral adjectives referring to the number of $(n-1)$-fold boundaries.

A point moving in one dimension traces a straight line. Keeping its extremities fixed, suppose the line broadened into an equilateral triangle by extension of its middle point in a direction perpendicular to the line. Again, suppose the middle point of the triangle extended along the third rectangular axis, carrying with it the plane so as to form a regular tetrahedron. Now let the middle point of the tetrahedron be extended in a direction perpendicular to the first three axes, carrying with it the four solid tetrahedra which extend from this central point to each tetrahedral face, until each of these becomes a regular tetrahedron. The enclosed four-fold figure is a penta-tessaract. In these processes one new angular point is added for each new dimension; a new line is

* Am. J. Math., Vol. 3, 1880, p. 6.

† "A New Era of Thought" (Swan, Sonnenschein & Co., London, 1888). For the literature of higher space see Am. J. Math., Vol. 1, 1878, pp. 261, 384 et seq., and Vol. 2, p. 65.

added for each point in the preceding figure; a new plane for each line, and so on; so that the law of increase of boundaries in this triangular series is very simple. See the table of figures following.

SERIES OF REGULAR FIGURES IN n-FOLD SPACE.

Series.	n	FIGURES.	NUMBER OF BOUNDARIES.						
			Points.	Lines.	Planes.	Solids.	Tessaracts.	Pentacts.	Hexacts.
Triangular.	1	Line, ...	2						
	2	Triangle, ...	3	3					
	3	Tetrahedron, ...	4	6	4				
	4	Penta-tessaract, ...	5	10	10	5			
	5	Hexa-pentact, ...	6	15	20	15	6		
	6	Hepta-hexact, ...	7	21	35	35	21	7	
	7	Okta-heptact, ...	8	28	56	70	56	28	8
		Etc.							
Rectangular.	1	Line, ...	2						
	2	Square, ...	4	4					
	3	Cube, ...	8	12	6				
	4	Tessaract, ...	16	32	24	8			
	5	Pentact, ...	32	80	80	40	10		
	6	Hexact, ...	64	192	240	160	60	12	
	7	Heptact, ...	128	448	672	560	280	84	14
		Etc.							
Oktahedral.	1	Line, ...	2						
	2	Square, ...	4	4					
	3	Oktahedron, ...	6	12	8				
	4	Hexadeka-tessaract, . .	8	24	32	16			
	5	Triakontaduo-pentact, .	10	40	80	80	32		
	6	Hexakontatetra-hexact,	12	60	160	240	192	64	
	7	128-heptact, ...	14	84	280	560	672	448	128
		Etc.							

Next suppose the line generated by a point to move along the second axis bodily, so as to generate a square. The square moving along the third axis generates a cube; and the cube, moving along the fourth axis, a rectangular tessaract (called for brevity simply a tessaract). In this series each point produces in the new figure a new line and point; each line a new square and line, each square a new cube and square, and so on according to this simple law of increase illustrated in the preceding table.

In picturing the generation of the oktahedral series we must suppose the point extended in two opposite directions to form a line; the middle (original) point of the line extended both ways along the second axis to form a square; the same point extended both ways along the third axis to form a regular oktahedron; along the fourth axis to form a hexadeka-tessaract; and so on. In this series two new angular points are added for each dimension, and each time the number of $(n-1)$-fold boundaries is doubled. The simplest law of the numbers of boundaries is found by observing that each figure of this series is the inverse of the corresponding figure of the rectangular series, and consequently each horizontal row of numbers in the oktahedral series in the preceding table must be the same as the corresponding row in the rectangular series taken in the reverse order. This conclusion may be easily confirmed by working out the slightly complex law of increase in the same way as for the other series.

I have selected the tessaract, the hexadeka-tessaract and the penta-tessaract as simple and typical illustrations of the synthetic method in projection, the simplicity and ease of which becomes more apparent as it is used.

2.—*Projection of a Square upon a Plane.*

The principles of parallel projection may be illustrated by considering briefly this case. Let a and b be the axes of the square, i. e. lines joining the middle points of opposite sides, and let N be a line through the center of the square and normal to it. Let P indicate the line of projection, which I shall take to be always perpendicular to the flat upon which the projection is made. The reader may follow more readily if he marks the axes on a square card, with a pin through its center to represent N.

Case I.—N parallel to P, and therefore a and b perpendicular to P.
The projection is an equal square.

Case II.—a parallel to P.

a becomes a point in the projection; b is unchanged; and the projected figure is therefore a line of unit length.

Case III.—N and a inclined, b perpendicular to P.

a is shortened; b and the angles at its extremities are unchanged. The projection is therefore a rectangle of unit length whose width may vary from zero to unity as limits.

Case IV.—N perpendicular, a and b inclined to P.

The two axes are projected upon one straight line. The figure is therefore a line whose length is between the limits unity and the square root of two.

Case V.—N, a and b inclined to P.

Both axes are shortened and the angle between them altered in the projection. The figure is that of an oblique parallelogram whose limits are three, a square, its diagonal on the one hand, and its side on the other hand.

The principles illustrated in this simple example may be summarized thus:

1. Parallel lines remain parallel.
2. Lines and angles perpendicular to P are unchanged.
3. Lines are shortened and angles altered when not perpendicular to P.
4. Lines parallel to P are reduced to zero, and angles to zero or 180° when in a plane parallel to P.

3.—*Projection of a Cube upon a Three-flat.*

Assuming now that there is a space of four dimensions, the assumption implies that these principles of projection, as well as all geometric relations that are true of three mutually perpendicular axes of three-fold space, are also true for any three of the four axes of four-fold space. Just as in our projection of the square it was necessary that the square should be outside of the plane upon which it was projected, so must our cube be placed outside of three-fold space in the direction of the fourth dimension, in order that it may be similarly projected upon our three-fold space. Let a, b, c be the three axes of this cube, i. e. lines joining the middle points of opposite faces; and let N be a line through the intersection of the axes, perpendicular to all of them. Let P be, as before, the line of projection, which is perpendicular to the three axes of the three-flat upon which the cube is to be projected. Now the essential difference between this

problem and our last lies in this, that instead of N we have now the third axis c, and have a new N whose location may be for the present inconceivable to us, but which bears the same relation to any one of the three known axes of the cube that these bear to each other.

Case I.—N parallel to P, and therefore a, b, c perpendicular to P.

The axes and angles are unaltered. The figure is a cube.

Case II.—a parallel to P, and therefore N, b and c perpendicular to P.

a becomes zero in the projection; b and c and the angles between them are unaltered. The figure is a square. It is evident also that whenever the cube is so situated that P passes through two points in it, or in other words, when P is in the same three-flat with the cube, the projected figure is plane; so that such cases need not be further considered here.

Case III.—N and a inclined, b and c perpendicular to P.

a is shortened; b and c and the angles between them and at their extremities are unaltered. The figure is therefore a right square prism varying between a square and a cube as limits.

Case IV.—N, a and b inclined, c perpendicular to P.

a and b are shortened and their inclination altered; c and the angles at its extremity are unchanged. The figure is therefore a vertical prism whose base is an oblique parallelogram. The sides of the parallelogram vary from zero to unity, and its greater diagonal from unity to the square root of two.

Case V.—N, a, b and c inclined to P.

All the axes are shortened and all the angles altered. The figure is a doubly oblique parallelopipedon whose limits are all of the figures of the four preceding cases.

The projection of any other regular solid upon a three-flat may be found in a similar way, or it may, in many cases, be inferred from the corresponding projection of the cube by suitable modifications of its boundaries.

4.—*Projection of a Tessaract.*

A tessaract is a four-fold figure bounded by eight cubes, one at each end of its four rectangular axes. These cubes are bounded by twenty-four squares, each of which is a double boundary; the squares by thirty-two lines, each of

which is common to three squares; the lines by sixteen points, each of which is the extremity of four lines. It must be noted that the twenty-four squares do not enclose the tessaract, but appear here and there upon its boundaries as edges do upon a cube; for a tessaract cannot be enclosed by planes any more than a cube can be enclosed by lines. In a three-flat projection, however, the tessaractic content disappears and the projection is bounded by planes, just as the plane projection of a solid is bounded by lines. The form of the faces bounding the projection of a tessaract may be found by consideration of the projection of the faces alone, but they are all determined by the positions and lengths of the projected axes of the tessaract. A tessaract is simply a cube extended to unit distance in the fourth dimension—in the direction of N. When any of the three axes of one of the bounding cubes is parallel to P, that cube is projected as a square. With the same nomenclature as before, except that N is now replaced by the fourth axis d; and designating the nearer cubes at the extremities of a, b, c and d by a_1, b_1, c_1, d_1, and the more distant cubes by a_2, b_2, c_2, d_2, we proceed.

Case I.—a parallel to P, therefore b, c and d perpendicular to P.

a is zero in the projection; b, c, d, and the angles between them are unaltered. The figure is a cube; or, more exactly, two coincident cubes, a_1 and a_2, whose faces are also coincident with the six squares which are the projections of the remaining six cubes of the tessaract.

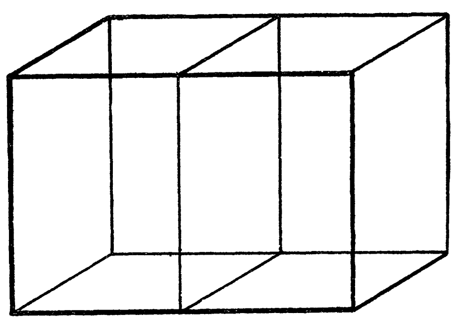

FIG. 1.

Case II.—a and b inclined, c and d perpendicular to P. (Fig. 1.)

a and b fall into a single line; c, d and the angles about them are unaltered. The figure is a right square prism, the length of whose $a + b$ axis is between unity and $\sqrt{2}$, consisting of the projections of a_1 and b_1 side by side, and coinci-

dent with these the pair b_2 and a_2. When a and b are equally inclined, a_2 coincides with b_1, and b_2 with a_1.

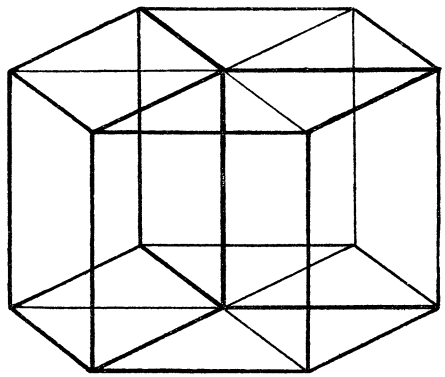

FIG. 2.

Case III.—a, b and c inclined; d perpendicular to P. (Fig. 2.)

a, b and c are shortened and the angles between them altered; d and the angles about it are unchanged. Since d remains perpendicular to all the other axes, a, b and c are in one plane in the projection. The figure is therefore a vertical hexagonal prism, which is regular when a, b and c are equally inclined to P, and in this case consists of a_1, b_1, c_1, coincident with a_2, b_2, c_2 in such a manner that a_2 coincides with the figure formed by the adjacent halves of b_1, c_1; and so on. When the hexagonal prism is not regular it varies toward a cube on one hand and toward the prism of *Case II* on the other hand.

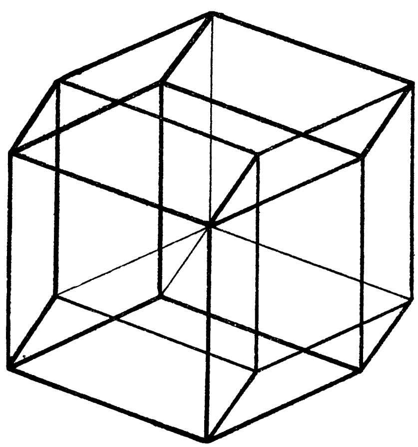

FIG. 3.

Case IV.—All the axes inclined to P. (Fig. 3.)

All the axes are shortened and there are no right angles in the projection. The eight cubes appear as two coincident sets of four. When all the axes are equally inclined to P each makes equal angles with the other three in the projection, that is to say, the common cube-edges, which are parallel to the four axes, diverge toward the four corners of a regular tetrahedron. Each of the four cubes, a_1, b_1, c_1, d_1, is in contact with each of the other three by one of its faces, leaving twelve faces external, bounding the figure, a rhombic dodekahedron. Coincident with this is another dodekahedron formed by the other four projected cubes in such a way that a_2 is coincident with the figure formed of the adjacent thirds of b_1, c_1 and d_1, and so on.

Irregular figures of this case are dodekahedra varying toward any of the figures of the first three cases.

Outline models, of wire, can be readily constructed to show the six projected cubes in *Case III*, and the eight in *Case IV*; though it is not easy to see them all in pictures of the models. (See figures.)

Plane projections of a tessaract are identical with plane projections of three-flat projections of the same. It is evident also that three-flat projections of figures in space of five or more dimensions are identical with three-flat projections of certain four-fold figures; so that no visualized conception of five-fold figures appears possible until four-fold conceptions become familiar to us.

An interesting projection of a tessaract is the well-known figure which corresponds to the perspective view of the interior of a cube as seen through its front face. This consists of a large outline cube having a smaller cube suspended at its center by eight strings connecting the eight pairs of adjacent corners. The analogy between the eight cubes thus represented in perspective and the six faces in the perspective plane projection of a cube is complete.

5.—*Projection of a Hexadeka-tessaract.*

A hexadeka-tessaract is a four-fold figure bounded by sixteen tetrahedra whose faces, as in the boundaries of all four-fold figures, are not free but between two solids. These thirty-two triangular faces are bounded by twenty-four lines, each line being common to four faces. From each of the eight angular points six lines pass to all the other points except the opposite extremity of the same axis. Since these eight points are situated at the ends of four equal and mutually perpendicular axes, the three-flat projections are easily outlined by joining

the extremities of the projected axes as found in the projection of the tessaract. I give here simply the resulting figures under the different cases, and designate by a_1, a_2, the extremities of the axis a, etc.

Case I.—a parallel to P.

a_1 and a_2 are coincident at the intersection of the axes. The figure is a regular oktahedron composed of eight tetrahedra which extend from its center to each of its faces.

Case II.—a and b inclined to P.

An oktahedron whose axes are 1, 1, $\frac{1}{2}\sqrt{2}$ is the figure when a and b are equally inclined. In other cases as the short axis approaches unity the points a_1, a_2, which are connected by lines to all the remaining angular points, move symmetrically from the extremities toward the center of the short axis.

Case III.—a, b and c inclined to P.

The typical form is that of two hexagonal prisms, base to base, having its vertical axis longer than the three horizontal axes, and enclosing twelve tetrahedra. This form varies toward that of *Case I* by the contraction of one of the horizontal axes toward its center while the other two extend toward unity in length and approach perpendicularity. It varies toward the regular form of *Case II* when two of the horizontal axes approach coincidence in direction and unity in length. The mode of variation toward the other forms of *Case II* is evident.

Case IV.—All the axes inclined to P.

When the axes are equally inclined the resulting figure is a cube, each of whose faces is crossed by two diagonal lines which complete the outlines of the sixteen tetrahedra included in the cube. The variations from this toward the forms of *Cases I*, *II* and *III* are readily seen.

6.—*Rotation.*

The only rotation possible in a plane is rotation about a point. In threefold space rotation about a point is also rotation about a line. Rotation is essentially motion in a plane, and when another dimension is added to the rotating body, another dimension is added also to the axis of rotation. In fourfold space, accordingly, every rotation takes place about a fixed axial plane. Rotation implies the motion of only two rectangular axes. All other axes perpendicular to these are not affected by it. Of the six mutually perpendicular

planes of a tessaract when rotation takes place in one, one other remains fixed and the other four move in a manner analogous to that of the two which pass through the fixed axis of a rotating cube.

The meaning of rotation about a plane becomes clearer when we consider its projection. In two-fold space the projection upon a line of a line rotating about one end is a line whose other extremity has a simple harmonic motion across the fixed point. In three-fold space the plane-projection of a rectangle rotating about one of its sides is (*a*) a line rotating about one extremity, (*b*) a rectangle, one side of which executes a simple harmonic motion across the opposite side which remains fixed, or (*c*) a parallelogram, one of whose sides moves elliptically about the opposite side. In four-fold space the three-flat projection of a cube rotating about one of its faces is (*a*) a square rotating about one of its sides, (*b*) a cube, one of whose faces executes a simple harmonic motion through the opposite face, or (*c*) a parallelopipedon, one of whose faces moves elliptically about the opposite face, so that twice during each rotation the parallelopipedon may become a plane. Rotation of a tessaract when the axial plane is perpendicular to P takes the form (*b*) in projection, and may be well illustrated in models constructed of wire and having hinge-joints. I have constructed such a model to show the change of Fig. 3 into Fig. 2, and conversely, as the tessaract is rotated. One of the four diagonals in Fig. 3 is made in two parts which telescope. Its upper half carries with it the ends of the three boundary wires attached to its extremity, and also the lower halves of the three diagonals parallel to them, so that as it sinks into the lower half of the diagonal the fourth cube is reduced to a plane. In other words, two coincident middle points in Fig. 3, each carrying three of the half-diagonals, move apart along the fourth diagonal, until they reach the middle points of its halves, where they coincide each with one of the extreme points of the same diagonal which have moved along to meet them.

The only transformations (apparent) in the physical world which correspond to rotation about a plane are, so far as I know, the formation of reflected images. With a thin shell we may rudely imitate the change. A glove rotated about a plane would fit the other hand, and we imitate the change by turning the glove inside out. True plane-rotation would reverse it, leaving it right side out.

7.—*Three-flat Projection of a Tetrahedron.*

Let N be a normal to any three rectangular axes of the tetrahedron; and suppose the tetrahedron so placed in four-fold space that N is parallel to P, the

line of projection. In this position the projection is exactly like its original. As the tetrahedron rotates about the plane of N and any axis a, its projection rotates about a. On the other hand when N and a are in the plane of rotation, the plane of b and c is fixed while any point of a executes a harmonic oscillation through that plane in the projection. It is evident then that any three-flat projection of a three-fold body may be obtained by supposing the body compressed along any desired axis, the limit of compression being such as will produce the reflected image of the original. This principle holds obviously for n-fold space.

An interesting case is the projection of a three-fold body which rotates about the plane containing any axis a, together with n, a line perpendicular to a and making an angle less than 90° with N. When the angle Nn is 45°, each point in the projection describes an ellipse in such a manner that, as the body makes one revolution about the fixed plane, the projection completes one revolution about a, and also, while rotating, has become gradually contracted along a fixed axis perpendicular to the fixed plane until wholly in that plane, and then returned to its original form. When the angle Nn is less than 45° the contraction does not proceed so far. When it is greater than 45°, the reverse figure is formed and the projection is a plane-figure twice during each revolution.

8.—*Projection of a Pentatessaract.*

In describing the triangular series of figures we supposed the pentatessaract derived from the tetrahedron by an extension of its middle point along the fourth axis. Suppose the pentatessaract placed so that P passes through this last point, d_1, and along the fourth axis, d, to the center of the original tetrahedron. Evidently the projection is the original tetrahedron with four others extending from its center to its four faces. Rotation about a plane containing d and any other axis, a, is projected as rotation about a. Rotation about any axial plane not containing d leaves that plane unaltered in the projection; while, as the figure rotates in the plane of d and any other axis, a; d is projected in the same straight line with a; and as d is extended a is shortened in the projection. Any projection of the pentatessaract may therefore be obtained by moving the point d_1 from the center of the tetrahedron along the line of any axis, a, while at the same time the tetrahedron is compressed proportionately along the axis a. The point d_1 of course remains connected with each of the other four angular points, marking out, in general, the other four tetrahedra.

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